

On the infimum of certain functionals

BIAGIO RICCERI

Here and in what follows, X is a real Banach space, $\varphi : X \rightarrow \mathbf{R}$ is a non-zero continuous linear functional and $\psi : X \rightarrow \mathbf{R}$ is a non-constant Lipschitzian functional with Lipschitz constant L .

From Proposition 2.1 of [2], we know that, when $L < \|\varphi\|_{X^*}$, the functional $\varphi + \psi$ is unbounded below. When, to the contrary, $L \geq \|\varphi\|_{X^*}$, this is no longer true. That is, the same functional can be bounded below. The simplest examples are provided by taking $\psi(x) = |\varphi(x)|$ or $\psi(x) = \|\varphi\|_{X^*}\|x\|$.

The aim of this very short paper is to study the infimum of that functional just when $L = \|\varphi\|_{X^*}$.

So, from now on, we assume that

$$L = \|\varphi\|_{X^*}.$$

Our basic result is as follows:

THEOREM 1. - *Let $[a, b]$ be a closed interval contained in $[-1, 1]$ and let $\gamma : [a, b] \rightarrow \mathbf{R}$ be a convex and lower semicontinuous function.*

Then, one has

$$\max \left\{ \inf_{x \in X} (\varphi(x) + a\psi(x)) - \gamma(a), \inf_{x \in X} (\varphi(x) + b\psi(x)) - \gamma(b) \right\} = \inf_{x \in X} \sup_{\lambda \in [a, b]} (\varphi(x) + \lambda\psi(x) - \gamma(\lambda)).$$

Our proof of Theorem 1 is based on the use of the following result (Theorem 5.9 and Remark 5.10 of [3]):

THEOREM A. - *Let S be a topological space, $I \subset \mathbf{R}$ a compact interval and $f : S \times I \rightarrow \mathbf{R}$ a function which is lower semicontinuous in X and upper semicontinuous and quasi-concave in I . Moreover, assume that there exists a set $D \subseteq I$ dense in I such that, for each $\lambda \in D$ and $r \in \mathbf{R}$, the set*

$$\{x \in S : f(x, \lambda) < r\}$$

is connected.

Then, one has

$$\sup_{\lambda \in I} \inf_{x \in S} f(x, \lambda) = \inf_{x \in S} \sup_{\lambda \in I} f(x, \lambda).$$

To be able to use Theorem A, we first have to establish the following result:

THEOREM 2. - *For each $\lambda \in]-1, 1[$ and $r \in \mathbf{R}$, the set $(\varphi + \lambda\psi)^{-1}(]-\infty, r])$ is a retract of X .*

PROOF. First, consider the multifunction $G : \mathbf{R} \rightarrow 2^X$ defined by

$$G(t) = \varphi^{-1}(]-\infty, t])$$

for all $t \in \mathbf{R}$. Let us check that

$$d_H(G(t), G(s)) \leq \frac{|t - s|}{\|\varphi\|_{X^*}}. \quad (1)$$

for all $t, s \in \mathbf{R}$, d_H being the usual Hausdorff distance. For instance, assume that $t < s$. Consequently

$$G(t) \subseteq G(s). \quad (2)$$

Now, fix $x \in G(s) \setminus G(t)$. Consequently

$$t < \varphi(x) \leq s .$$

In view of the classical formula giving the distance of a point from a closed hyperplane, we have

$$\text{dist}(x, G(t)) \leq \text{dist}(x, \varphi^{-1}(t)) = \frac{\varphi(x) - t}{\|\varphi\|_{X^*}} \leq \frac{s - t}{\|\varphi\|_{X^*}} .$$

So

$$\sup_{x \in G(s)} \text{dist}(x, G(t)) \leq \frac{s - t}{\|\varphi\|_{X^*}}$$

which together with (2) gives (1). Now, consider the multifunction $F : X \rightarrow 2^X$ defined by

$$F(x) = G(r - \lambda\psi(x))$$

for all $x \in X$. For each $x, y \in X$, we have

$$d_H(F(x), F(y)) \leq \frac{1}{\|\varphi\|_{X^*}} |\lambda| |\psi(x) - \psi(y)| \leq |\lambda| \|x - y\| .$$

Hence, since $|\lambda| < 1$, F is a multivalued contraction with closed and convex values. Then, in view of [1], the set $\text{Fix}(F) := \{x \in X : x \in F(x)\}$ is a retract of X . To complete the proof, simply observe that $\text{Fix}(F) = (\varphi + \lambda\psi)^{-1}[-\infty, r]$. \triangle

Proof of Theorem 1. Consider the function $f : [a, b] \rightarrow \mathbf{R}$ defined by

$$f(x, \lambda) = \varphi(x) + \lambda\psi(x) - \gamma(\lambda)$$

for all $(x, \lambda) \in X \times [a, b]$. Clearly, f is continuous in X , while it is upper semicontinuous and concave in $[a, b]$. Fix $\lambda \in]a, b[$ and $r \in \mathbf{R}$. Of course, we have

$$\{x \in X : f(x, \lambda) < r\} = \bigcup_{s < r} \{x \in X : f(x, \lambda) \leq s\} .$$

On the other hand, by Theorem 2, the sets of the family $\{\{x \in X : f(x, \lambda) \leq s\}\}_{s < r}$ are connected (being retracts of X) and pairwise non-disjoint. Consequently, the set $\{x \in X : f(x, \lambda) < r\}$ is connected too. Therefore, we can apply Theorem A. It ensures that

$$\sup_{\lambda \in [a, b]} \inf_{x \in X} (\varphi(x) + \lambda\psi(x) - \gamma(\lambda)) = \inf_{x \in X} \sup_{\lambda \in [a, b]} (\varphi(x) + \lambda\psi(x) - \gamma(\lambda)) .$$

Now, observe that, since $\inf_{x \in X} (\varphi(x) + \lambda\psi(x)) = -\infty$ for all $\lambda \in]-1, 1[$, we have

$$\sup_{\lambda \in [a, b]} \inf_{x \in X} (\varphi(x) + \lambda\psi(x) - \gamma(\lambda)) = \max \left\{ \inf_{x \in X} (\varphi(x) + a\psi(x)) - \gamma(a), \inf_{x \in X} (\varphi(x) + b\psi(x)) - \gamma(b) \right\}$$

and the conclusion follows. \triangle

A consequence of Theorem 1 is as follows:

THEOREM 3. - *Let $[a, b]$ be a closed interval contained in $[-1, 1]$ and let $\gamma : [a, b] \rightarrow \mathbf{R}$ be a continuous function which is derivable in $]a, b[$. Assume that γ' is strictly increasing in $]a, b[$. Set*

$$A = \left\{ x \in X : \psi(x) \leq \inf_{]a, b[} \gamma' \right\} ,$$

$$B = \left\{ x \in X : \psi(x) \geq \sup_{]a, b[} \gamma' \right\}$$

and

$$C = \left\{ x \in X : \inf_{]a, b[} \gamma' < \psi(x) < \sup_{]a, b[} \gamma' \right\} .$$

Finally, denote by η the inverse of the function γ' .

Then, one has

$$\begin{aligned} & \max \left\{ \inf_{x \in X} (\varphi(x) + a\psi(x)) - \gamma(a), \inf_{x \in X} (\varphi(x) + b\psi(x)) - \gamma(b) \right\} = \\ & \min \left\{ \inf_{x \in A} (\varphi(x) + a\psi(x)) - \gamma(a), \inf_{x \in B} (\varphi(x) + b\psi(x)) - \gamma(b), \inf_{x \in C} (\varphi(x) + \eta(\psi(x))\psi(x) - \gamma(\eta(\psi(x)))) \right\} . \end{aligned}$$

PROOF. Let f be as in the proof of Theorem 1. Fix $x \in X$. Clearly, $f(x, \cdot)$ is concave in $[a, b]$. Moreover, according to the sign of its derivative, the function $f(x, \cdot)$ is non-increasing (resp. non-decreasing) in $[a, b]$ if $x \in A$ (resp. $x \in B$). If $x \in C$, the derivative of $f(x, \cdot)$ vanishes at the point $\eta(\psi(x))$ and so, by concavity, such a point is the global maximum of $f(x, \cdot)$ in $[a, b]$. Summarizing, we have

$$\sup_{\lambda \in [a, b]} f(x, \lambda) = \begin{cases} \varphi(x) + a\psi(x) - \gamma(a) & \text{if } x \in A \\ \varphi(x) + b\psi(x) - \gamma(b) & \text{if } x \in B \\ \varphi(x) + \eta(\psi(x))\psi(x) - \gamma(\eta(\psi(x))) & \text{if } x \in C , \end{cases}$$

and the conclusion clearly follows in view of Theorem 1. \triangle

In turn, applying Theorem 3, we obtain the following result:

THEOREM 4. - *We have*

$$\max \left\{ \inf_{x \in X} (\varphi(x) + \psi(x)), \inf_{x \in X} (\varphi(x) - \psi(x)) \right\} = \inf_{x \in X} (\varphi(x) + |\psi(x)|) \quad (3)$$

and

$$\liminf_{\|x\| \rightarrow +\infty} (\varphi(x) + |\psi(x)|) = \inf_{x \in X} (\varphi(x) + |\psi(x)|) . \quad (4)$$

PROOF. First, we want to prove that

$$\max \left\{ \inf_{x \in X} (\varphi(x) + \psi(x)), \inf_{x \in X} (\varphi(x) - \psi(x)) \right\} = \inf_{x \in X} (\varphi(x) + |\psi(x)| + e^{-|\psi(x)|}) . \quad (5)$$

Consider the function $\gamma : [-1, 1] \rightarrow \mathbf{R}$ defined by

$$\gamma(\lambda) = \begin{cases} (1 - |\lambda|) \log(1 - |\lambda|) + |\lambda| & \text{if } |\lambda| < 1 \\ 1 & \text{if } |\lambda| = 1 . \end{cases}$$

Clearly, γ is continuous in $[-1, 1]$, is derivable in $] - 1, 1[$, γ' is strictly increasing and $\gamma'([-1, 1]) = \mathbf{R}$. Moreover, η , the inverse of γ' , is given by

$$\eta(\mu) = \begin{cases} \frac{|\mu|}{\mu} (1 - e^{-|\mu|}) & \text{if } \mu \neq 0 \\ 0 & \text{if } \mu = 0 . \end{cases}$$

So, for each $x \in X \setminus \psi^{-1}(0)$, we have

$$\eta(\psi(x))\psi(x) - \gamma(\eta(\psi(x))) = |\psi(x)|(1 - e^{-|\psi(x)|}) - (-e^{-|\psi(x)|}|\psi(x)| + 1 - e^{-|\psi(x)|}) = |\psi(x)| + e^{-|\psi(x)|} - 1 .$$

Clearly, these equalities hold also if $\psi(x) = 0$. Consequently, by Theorem 2, after observing that $C = X$, we have

$$\begin{aligned} \max \left\{ \inf_{x \in X} (\varphi(x) - \psi(x)) - 1, \inf_{x \in X} (\varphi(x) + \psi(x)) - 1 \right\} &= \inf_{x \in X} (\varphi(x) + \eta(\psi(x))\psi(x)) - \gamma(\eta(\psi(x))) \\ &= \inf_{x \in X} (\varphi(x) + |\psi(x)| + e^{-|\psi(x)|}) - 1 \end{aligned}$$

which yields (5). Since

$$\max \left\{ \inf_{x \in X} (\varphi(x) + \psi(x)), \inf_{x \in X} (\varphi(x) - \psi(x)) \right\} \leq \inf_{x \in X} (\varphi(x) + |\psi(x)|) \leq \inf_{x \in X} (\varphi(x) + |\psi(x)| + e^{-|\psi(x)|}) ,$$

from (5), we obtain both (3) and

$$\inf_{x \in X} (\varphi(x) + |\psi(x)|) = \inf_{x \in X} (\varphi(x) + |\psi(x)| + e^{-|\psi(x)|}) . \quad (6)$$

Finally, let us prove (4). Arguing by contradiction, assume that

$$\inf_{x \in X} (\varphi(x) + |\psi(x)|) < \liminf_{\|x\| \rightarrow +\infty} (\varphi(x) + |\psi(x)|) .$$

Fix ξ satisfying

$$\inf_{x \in X} (\varphi(x) + |\psi(x)|) < \xi < \liminf_{\|x\| \rightarrow +\infty} (\varphi(x) + |\psi(x)|) . \quad (7)$$

So, there is some $\delta > 0$ such that

$$\varphi(x) + |\psi(x)| > \xi \quad (8)$$

for all $x \in X$ satisfying $\|x\| > \delta$. Now, in view of (6), we can fix a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} (\varphi(x_n) + |\psi(x_n)| + e^{-|\psi(x_n)|}) = \inf_{x \in X} (\varphi(x) + |\psi(x)|) . \quad (9)$$

Clearly

$$\lim_{n \rightarrow \infty} (\varphi(x_n) + |\psi(x_n)|) = \inf_{x \in X} (\varphi(x) + |\psi(x)|) . \quad (10)$$

In view of (7), there is $\nu \in \mathbf{N}$ such that

$$\varphi(x_n) + |\psi(x_n)| < \xi$$

for all $n > \nu$. Thus, by (8), we have

$$\sup_{n > \nu} \|x_n\| \leq \delta .$$

Then, since ψ is Lipschitzian, the sequence $\{\psi(x_n)\}$ is bounded too. But, (9) and (10) imply that

$$\lim_{n \rightarrow \infty} e^{-|\psi(x_n)|} = 0$$

which leads to a contradiction. The proof is complete. \triangle

We conclude with a consequence of Theorem 3.

PROPOSITION 1. - Assume that ψ is Gâteaux differentiable and that both φ and $-\varphi$ do not belong to $\psi'(X)$.

Then, for every $r \in \mathbf{R}$, the functional $x \rightarrow \varphi(x) + |\psi(x) - r|$ has no global minima in X .

PROOF. Arguing by contradiction, assume that there is $x_0 \in X$ such that

$$\varphi(x_0) + |\psi(x_0) - r| = \inf_{x \in X} (\varphi(x) + |\psi(x) - r|) .$$

Then, by Theorem 3 (applied to $\psi - r$), x_0 would be a global minimum either of $\varphi + \psi$ or of $\varphi - \psi$. Accordingly, we would have either $\psi'(x_0) = -\varphi$ or $\psi'(x_0) = \varphi$, contrary our assumption. \triangle

REMARK 1. - Of course, if $\|\psi'(x)\|_{X^*} < L$ for all $x \in X$, then both φ and $-\varphi$ do not belong to $\psi'(X)$.

References

- [1] B. RICCERI, *Une propriété topologique de l'ensemble des points fixes d'une contraction multivoques à valeurs convexes*, Rend. Accad. Naz. Lincei, (8) **81** (1987), 283-286.
- [2] B. RICCERI, *Further considerations on a variational property of integral functionals*, J. Optim. Theory Appl., **106** (2000), 677-681.
- [3] B. RICCERI, *Nonlinear eigenvalue problems*, in “Handbook of Nonconvex Analysis and Applications” D. Y. Gao and D. Motreanu eds., 543-595, International Press, 2010.

Department of Mathematics
University of Catania
Viale A. Doria 6
95125 Catania, Italy
e-mail address: ricceri@dmi.unict.it